

# Unification of Dark Matter and Dark Energy: the Inhomogeneous Chaplygin Gas

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## Abstract

We extend the world model of Kamenshchik et al. to large perturbations by formulating a Zeldovich-like approximation. We sketch how this model unifies dark matter with dark energy in a geometric setting reminiscent of  $M$ -theory.

After nearly two decades of reign [1], the Einstein-de Sitter dust model has been swept aside by observations of high redshift supernovae [2] which suggest that the Hubble expansion is accelerating. When combined with the Boomerang/Maxima data [3] showing that the location of the first acoustic peak in the power spectrum of the microwave background is consistent with the inflationary prediction  $\Omega = 1$ , the evidence for a net equation of state of the cosmic fluid lying in the range  $-1 \leq w = P/\rho < -1/3$  is compelling. Parametrically,  $w = P_{\text{DE}}/(\rho_{\text{DE}} + \rho_{\text{DM}}) = -\Omega_{\Lambda}$  gives a ratio of unclustered dark energy to clustered dark matter of order 7:3, thereby also resolving the longstanding  $\Omega_{\text{DM}} < 1$  puzzle [1, 4] implied by peculiar velocity fields. The theoretical implications of these discoveries are profound. Simply appending a

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nonzero cosmological constant  $\Lambda$  to standard cold dark matter ( $\Lambda$ CDM, [4]) invites anthropic arguments [5] as to why both  $\Omega_{\text{DM}}$  and  $\Omega_{\Lambda}$  are of order unity today. False vacuum models have been proposed [6], and, in one formulation [7], linked to quantum gravity effects and axionic CDM.

Currently, the most popular alternative is Quintessence [8] which conventionally entails a real, homogeneous scalar field of mass  $m_Q < H_0$  whose potential is arranged such that it silently tracks radiation and matter, gracefully entering a dominant de Sitter phase with an intermediate mixed epoch (QCDM) today. Acceptable models can be obtained using pseudo-Goldstone bosons [9], while spintessence [10] is a twist on this theme involving a complex quintessence field.

If  $\Lambda$ CDM appears too coincidental, QCDM needs two distinct fields, one to describe dark matter, the other dark energy. Economy would suggest that dark energy and dark matter should be different manifestations of the same entity. Wetterich [11] has speculated that dark matter may be cosmon (quintessence) lumps while Kasuya [12] has pointed out that spintessence-like models are generally unstable to formation of Q-balls which behave as pressureless matter. Because they rely upon the coupled nonlinear scalar-gravitational fields even into the nonlinear regime, it is difficult to assess these proposals in the absence of large-scale numerical simulations. This is in marked contrast to  $\Omega_{\text{CDM}} = 1$  where the Zeldovich approximation [13] allows one to obtain an intuitive picture. Being essentially a variant of  $\Lambda$ CDM, the Barr-Sechel scenario [7] does not suffer this criticism.

In a recent paper Kamenshchik, Moschella and Pasquier [14] have studied a homogeneous model based on a single fluid obeying the Chaplygin gas equation of state

$$P = -A/\rho, \quad (1)$$

which has been intensively investigated for its solubility in 1+1-dimensional space-time, its supersymmetric extension and connection to d-branes [15]. Implementing this in the relativistic energy conservation equation, the density evolves according to

$$\rho(a) = \sqrt{A + B/a^6}, \quad (2)$$

where  $a$  is the scale factor and  $B$  an integration constant. This model smoothly interpolates between dust and de Sitter phases without ad hoc

assumption. Fabris, Goncalves and de Souza [16] have endeavoured to examine density perturbations to this model, but, owing to an unfortunate choice of the lightcone gauge, their unperturbed Newtonian equations cannot reproduce (2), which is a prerequisite. In any case, the universe is very inhomogeneous [17] today, and if one takes this idea seriously one needs a Zeldovich-like nonperturbative approach adopted to the case  $P \neq 0$ .

The purpose of this letter is to give such a formulation covariantly and in sufficient generality that it can be adopted to any barometric or parametric equation of state. Consider the Lagrangian

$$\mathcal{L} = g^{\mu\nu} \Phi^*_{,\mu} \Phi_{,\nu} - V(|\Phi|^2) \quad (3)$$

for a complex scalar field  $\Phi = (\phi/\sqrt{2}m) \exp(-im\theta)$ , where  $m$  is the mass appearing in the potential  $V$ . The Lagrangian (3), expressed in terms of  $\phi$  and  $\theta$ , reads

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \left( \phi^2 \theta_{,\mu} \theta_{,\nu} + \frac{1}{m^2} \phi_{,\mu} \phi_{,\nu} \right) - V(\phi^2/2). \quad (4)$$

We now apply the Thomas-Fermi approximation [18]. In contrast to Kamenishchik et al. [14] who assumed spatial homogeneity, we allow for space-time variations of  $\phi$  on scales larger than  $m^{-1}$ . More precisely, we assume that

$$\phi_{,\mu} \ll m\phi, \quad (5)$$

and hence, we may neglect the second term in the brackets on the right hand side of Eq. (4). This yields the Lagrangian

$$\mathcal{L}_{\text{TF}} = \frac{\phi^2}{2} g^{\mu\nu} \theta_{,\mu} \theta_{,\nu} - V(\phi^2/2) \quad (6)$$

and the equations of motion for the fields  $\phi$  and  $\theta$

$$g^{\mu\nu} \theta_{,\mu} \theta_{,\nu} = V'(\phi^2/2), \quad (7)$$

$$(\sqrt{-g} \phi^2 g^{\mu\nu} \theta_{,\nu})_{,\mu} = 0, \quad (8)$$

where  $V'(x) = dV/dx$ . Assuming  $V' > 0$ , the field  $\theta$  may be treated as a velocity potential for the fluid 4-velocity

$$U^\mu = g^{\mu\nu} \theta_{,\nu} / \sqrt{V'}, \quad (9)$$

that is restricted to the mass shell, i.e.,  $U_\mu U^\mu = 1$ . As a consequence, the stress-energy tensor  $T^{\mu\nu}$  constructed from the Lagrangian (6) takes the perfect fluid form, with the parametric equation of state

$$\rho = \frac{\phi^2}{2}V' + V, \quad P = \frac{\phi^2}{2}V' - V. \quad (10)$$

This procedure may be reversed as in the case of Newtonian limit [19]. Suppose the equation of state is given, e.g., in parametric form  $P(\psi), \rho(\psi)$ . From Eqs. (10) we find

$$\ln(\phi^2) = \int \frac{d\rho - dP}{\rho + P}. \quad (11)$$

Using this one obtains  $\rho(\phi^2)$  or  $\psi(\phi^2)$  and hence the potential

$$V = \frac{1}{2}(\rho - P) \quad (12)$$

of the field theory associated with the fluid, the integration constant being fixed by the  $P = 0$  limit. For the Chaplygin gas one finds  $\rho = \phi^2$  and

$$V = \frac{1}{2} \left( \phi^2 + \frac{A}{\phi^2} \right). \quad (13)$$

We remark that this potential is very different from that of Kamenshchik et al. [14] whose derivation is based on the assumption of homogeneity. Note that (11) has the structure of a renormalization group for the fluid-field map.

For a sensible theory with  $0 \geq w \geq -1$  and a speed of sound satisfying  $0 \leq C_s^2 = dP/d\rho \leq 1$ , the relativistic limits of these inequalities should coincide, which uniquely selects (1). It is interesting to note that  $P = -A/\rho^\alpha$  gives the functional flow

$$\phi^2(\rho) = \rho^\alpha(\rho^{\alpha+1} - A)^{\frac{1-\alpha}{1+\alpha}} \quad (14)$$

which, as  $\rho^{\alpha+1} - A \rightarrow 0^+$ , resembles that of the periodic gaussian model [20]. The critical line  $\phi^2 = \rho$  describes again the Chaplygin gas and  $V^* = (\phi^2 + A/\phi^2)/2$  is the corresponding critical potential. Introducing  $K = \phi^2/\sqrt{A}$  which measures the dimensionless coupling of the  $\theta$  field in (6), the potential  $V^*$  is self-dual under  $K \rightarrow K^{-1}$ , reminiscent of M-theory string dualities [21]. Indeed, the physical content of the full  $\mathcal{L}^*$  shares this duality, as can be

seen through eliminating  $\phi^2$  by its algebraic equation of motion. This yields a Born-Infeld theory [22]

$$\mathcal{L}^* = -\sqrt{A} \sqrt{1 - g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}} \quad (15)$$

describing space-time as a 3+1 brane in a 4+1 dimensional bulk [23]. The velocity potential  $\theta$  in (9) measures surface excursion in the fifth dimension in Eq. (15).

Notably, the Chaplygin gas appears in the stabilization of branes in Schwarzschild-AdS black hole bulks as a critical theory at the horizon [24]. Also, a free-energy similar to Eq. (13) occurs in the stringy analysis of black holes in three dimensions [25]. Replacing Rama's "string bits" with "brane cells", the free energy (13) follows on dimensional grounds,  $\phi^{-1}$  corresponding to the size of the system.

Of course the primary question is whether  $\mathcal{L}^*$  yields a reasonable inhomogeneous cosmology. For this purpose we do not eliminate  $\phi$  from the Lagrangian (6) with (13), writing the field equation (8) in the form

$$\left( \sqrt{-g\rho(\rho + P)} U^\mu \right)_{,\mu} = 0 \quad (16)$$

instead. To solve this equation, it is convenient to use comoving coordinates. In the general comoving coordinate system the 4-velocity vector is given by  $U^\mu = \delta_0^\mu / \sqrt{g_{00}}$ . Eq. (16) then becomes

$$\left( \sqrt{\frac{-g}{g_{00}}} (\rho^2 - A) \right)_{,0} = 0, \quad (17)$$

with the solution

$$\frac{-g}{g_{00}} (\rho^2 - A) = B(\mathbf{x}), \quad (18)$$

where  $B$  is an arbitrary function independent of  $x^0$ . For a general metric  $g_{\mu\nu}$ , the proper time is  $d\tau = \sqrt{g_{00}} dx^0$  and  $-g/g_{00} \equiv \gamma$  is the determinant of the induced 3-metric

$$\gamma_{ij} = \frac{g_{i0}g_{j0}}{g_{00}} - g_{ij}. \quad (19)$$

Noting that for the relevant scales  $B(\mathbf{x})$  can be considered to be approximately constant, we obtain

$$\rho = \sqrt{A + \frac{B}{\gamma}} \quad (20)$$

as the generalization of Eq. (2), in synchronous coordinates ( $t = \tau, \mathbf{x}$ ). We stress here a peculiarity of negative versus ordinary positive pressure fluids: for the former, as the fluid becomes ultrarelativistic, the weak energy condition  $\rho + P \geq 0$  is saturated so that  $T^{\mu\nu}_{;\nu} = 0$  implies  $P_{;\mu} = 0$  and there are no pressure gradients. Thus in both the nonrelativistic and ultrarelativistic regimes the synchronous coordinates are comoving.

Based on this observation, we can take over from the dust case [26] the geometric implementation of the Zeldovich approximation as a map from Lagrange ( $\mathbf{q}$ ) to Euler (comoving-synchronous,  $\mathbf{x}$ ) coordinates inducing the 3-metric

$$\gamma_{ij} = \delta_{kl} D_i^k D_j^l. \quad (21)$$

Here  $D$  is the deformation tensor

$$D_i^j = a(t) \left( \delta_i^j - b(t) \frac{\partial^2 \varphi(\mathbf{q})}{\partial q^i \partial q_j} \right), \quad (22)$$

and  $\varphi$  the peculiar velocity potential. The function  $b$ , which describes the evolution of the density contrast, may be calculated in the standard way [27] treating the quantity

$$h = 2b(t)\varphi_{,i}^i \quad (23)$$

as a perturbation. From Eqs. (20)-(23) one can derive

$$\rho \simeq \bar{\rho}(1 + \delta), \quad P \simeq -\frac{A}{\bar{\rho}}(1 - \delta), \quad (24)$$

with

$$\bar{\rho} = \sqrt{A + B/a^6}, \quad \delta = \frac{h}{2}(1 + w), \quad (25)$$

where

$$w = -\frac{A}{\bar{\rho}^2}. \quad (26)$$

Using these equations and the synchronous metric defined in Eq. (21) with (22) we obtain the 0-0 component of the Einstein field equations in the form

$$-3\frac{\ddot{a}}{a} + \frac{1}{2}\ddot{h} + \frac{\dot{a}}{a}\dot{h} = 4\pi\bar{\rho}[(1 + 3w) + (1 - 3w)\delta]. \quad (27)$$

The unperturbed part of this equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\bar{\rho}(1 + 3w) \quad (28)$$

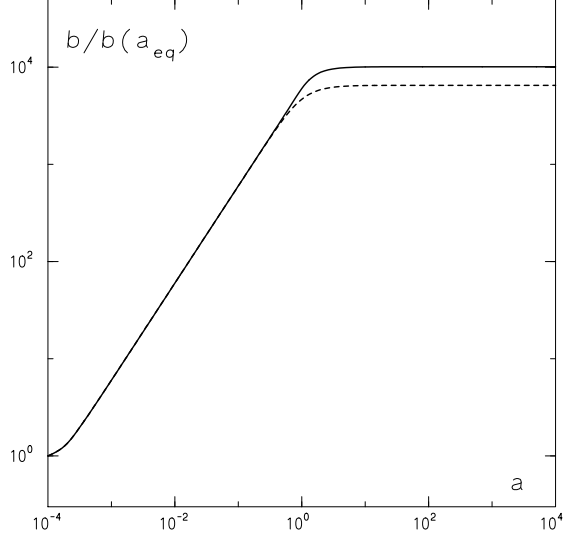


Figure 1: Evolution of  $b(a)/b(a_{\text{eq}})$  from  $a_{\text{eq}} = 1.0 \times 10^{-4}$  for  $\Omega_{\Lambda} = 0.7$  and  $b'(a_{\text{eq}}) = 0$ , for the Chaplygin gas (solid line) and  $\Lambda$ CDM (dashed line).

coincides with the second Friedmann equation. The first-order part of (27) may be written as a differential equation for  $b(a)$ , i.e.,

$$\frac{2}{3}a^2b'' + a(1-w)b' = (1+w)(1-3w)b, \quad (29)$$

where the prime denotes the derivative with respect to  $a$ . The function  $w(a)$  may be conveniently expressed in terms of the standard parameter  $\Omega_{\Lambda}$  which was introduced previously, i.e.,

$$w(a) = -\frac{\Omega_{\Lambda}a^6}{1 - \Omega_{\Lambda} + \Omega_{\Lambda}a^6}. \quad (30)$$

In Fig. 1 the evolution of  $b(a)$  is shown from radiation-matter equality with the initial condition  $b'(a_{\text{eq}}) = 0$ . Owing to both the high power of  $a$

appearing in Eq. (30) and the factor of  $(1 - 3w)$  in (29), the difference to the standard CDM scenario at  $a = 1$ , i.e., today, is negligible. Note that, while the metric perturbation  $h \propto b$  saturates, the density contrast  $\delta \propto (1 + w)b$  vanishes for large  $a$ . In the same figure we give the corresponding result for  $\Lambda$ CDM which is obtained by omitting the factor  $(1-3w)$  in Eq. (29) and changing  $a^6$  to  $a^3$  in Eq. (30); the contrast  $\delta$  in that case is 50% smaller at  $a = 1$ . Thus for the inhomogeneous Chaplygin gas we can take over the dust results bodily up to  $z \simeq 0$ . The picture which emerges is that on caustics, where galactic halos and clusters form, we have  $w \simeq 0$ , i.e., the fluid behaves as dark matter. Conversely, in the voids  $w \simeq -\Omega_\Lambda$  drives the acceleration as dark energy. Here the answer to the coincidence question mentioned earlier is that acceleration sets in only once the observed cellular structure develops.

In conclusion, we have shown that the inhomogeneous Chaplygin gas offers a simple unified model of dark matter and dark energy. It may be worth pointing out that the potential  $V$  obtained from the simple equation of state (1) may easily be generalized by adding a more complicated interaction term, e.g., a power of  $\phi$  greater than 2, and in this way generate a more refined equation of state for dark matter, instead of  $w = 0$ . This would not alter our general picture, as long as this additional interaction term is much smaller than  $\sqrt{A}$  when  $\phi^2$  approaches  $\sqrt{A}$ . In particular, one can add a thermal component to (1) which, although irrelevant to the large-scale structure and acceleration, provides a finite phase-space density inferred from galactic cores [28]. Ordinary matter can be included using Sundrum's [23] effective field theory; the  $\theta$  field is derivatively coupled and thus harmless. The fact that this unification is achieved in a geometric framework, having roots in the "theory of everything", makes this scenario all the more remarkable.

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## References



- [1] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).
- [2] S. Perlmutter et al., *Astrophys. J.* **517**, 565 (1998); A.G. Riess et al., *Astron. J.* **116**, 109 (1998).
- [3] P. de Bernardis et al., *Nature* **404**, 995 (2000); S. Hanany et al., *Astrophys. J.* **545**, L5 (2000), astro-ph/0005123.
- [4] P.J.E. Peebles, *Astrophys. J.* **284**, 439 (1984); J.P. Ostriker and P.J. Steinhardt, *Nature* **377**, 600 (1995).
- [5] J. Carriga and A. Vilenkin, hep-th/0011262; M. Dine, hep-th/0107259.
- [6] G. Dvali and A. Vilenkin, hep-th/0102142;
- [7] S.M. Barr and D. Seichel, hep-ph/0106239.
- [8] C. Wetterich, *Nucl. Phys. B* **302**, 668 (1988); P.J.E. Peebles and B. Ratra, *Astrophys. J.* **325**, L17 (1988); R.R. Caldwell, R. Dave and P.J. Steinhardt, *Phys. Rev. Lett.* **80** 1582 (1998); I. Zlatev, L. Wang and P.J. Steinhardt, *Phys. Rev. Lett.* **82**, 896 (1999).
- [9] K. Coble, S. Dodelson and J.A. Frieman, *Phys. Rev. D* **55**, 1851 (1997).
- [10] L.A. Boyle, R.R. Caldwell and M. Kamionkowski, astro-ph/0105318; J.-An Cu and W.-Y.P. Hwang, astro-ph/0105099.
- [11] C. Wetterich, hep-ph/0108266, astro-ph/0108411.
- [12] S. Kasuya, astro-ph/0105408.
- [13] Ya. B. Zeldovich, *Astron. Astrophys.* **5**, 84 (1970); for a review see V. Sahni and P. Coles, *Phys. Rep.* **262**, 1 (1995).
- [14] A. Kamenshchik, U. Moschella, and V. Pasquier, *Phys. Lett. B* **511**, 265 (2001).
- [15] Y. Bergner, R. Jackiw, *Phys. Lett. A* **284**, 146 (2001); for a review see R. Jackiw, physics/0010042.
- [16] J.C. Fabris, S.B.V. Goncalves and P.E. de Souza, gr-qc/0103083.

- [17] For a recent review see K.K.S. Wu, O. Lahar, and M. Rees, *Nature* **397**, 225 (1999).
- [18] The terminology borrows from Bose-Einstein condensates: A.S. Parkins and D.F. Walls, *Phys. Rep.* **303**, 1 (1998).
- [19] N. Bilic, R.J. Lindebaum, G.B. Tupper, and R.D. Viollier, *Phys. Lett. B* **515**, 105 (2001).
- [20] J.B. Kogut, *Rev. Mod. Phys.* **51**, 659 (1979).
- [21] J. Polchinski, *String Theory* (U. Cambridge Press, Cambridge, 1998).
- [22] R. Jackiw and A.P. Polychronakos, *Commun. Math. Phys.* **207**, 107 (1999); N. Ogawa, *Phys. Rev. D* **62**, 085023 (2000), and references therein.
- [23] R. Sundrum, *Phys. Rev. D* **59**, 085009 (1999); for a review of brane world models see V.A. Rubakov, hep-ph/0104152.
- [24] A. Kamenshchik, U. Moschella and V. Pasquier, *Phys. Lett. B* **487**, 7 (2000).
- [25] S.K. Rama, *Phys. Lett. B* **424**, 39 (1998).
- [26] S. Matarrese and D. Terranova, *Mon. Not. R. Astron. Soc.* **283**, 400 (1996).
- [27] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, NJ, 1993).
- [28] N. Bilic, G.B. Tupper, and R.D. Viollier, in preparation.